1 a

$$\begin{aligned} b &= aq + r,\ 0 \le r < a \\ 43 &= 5q + r \\ &= 5 \times 8 + 3 \end{aligned}$$

$$\begin{aligned} (5, 43) &= (5, 3) \text{ by theorem 2} \\ 5 &= 3 \times 1 + 2 \\ (3, 5) &= (3, 2) \text{ by theorem 2} \\ 3 &= 2 \times 1 + 1 \\ (2, 3) &= (2, 1) \text{ by theorem 2} \\ 2 &= 2 \times 1 + 0 \end{aligned}$$

$\therefore\quad (43, 5) = (5, 3) = 1$

b

$$\begin{aligned} b &= aq + r,\ 0 \le r < a \\ 39 &= 13q + r \\ &= 13 \times 3 + 0 \end{aligned}$$

$\therefore\quad (39, 13) = (13, 0) = 13$

c

$$\begin{aligned} b &= aq + r,\ 0 \le r < a \\ 37 &= 17q + r \\ &= 17 \times 2 + 3 \end{aligned}$$

$$\begin{aligned} (17, 37) &= (17, 3) \text{ by theorem 2} \\ 17 &= 3 \times 5 + 2 \\ (3, 17) &= (3, 2) \text{ by theorem 2} \\ 3 &= 2 \times 1 + 1 \\ (2, 3) &= (2, 1) \text{ by theorem 2} \\ 2 &= 2 \times 1 + 0 \end{aligned}$$

$\therefore\quad (37, 17) = (17, 3) = 1$

d

$$\begin{aligned} b &= aq + r,\ 0 \le r < a \\ 128 &= 16q + r \\ &= 16 \times 8 + 0 \end{aligned}$$

$\therefore\quad (128, 16) = (16, 0) = 16$

2 If d is a common factor of a and b, then $a = nd$ and $b = md$.

$$\begin{aligned} a + b &= nd + md \\ &= d(n + m) \end{aligned}$$

$\therefore\quad d$ is a divisor of $a + b$.

If d is a common factor of a and b, then $a = nd$ and $b = md$.

$$\begin{aligned} a - b &= nd - md \\ &= d(n - m) \end{aligned}$$

$\therefore\quad d$ is a divisor of $a - b$.

3 a

$$\begin{aligned} 9284 &= 4361 \times 2 + 562 \\ (4361, 9284) &= (4361, 562) \\ 4361 &= 562 \times 7 + 427 \\ (4361, 562) &= (427, 562) \\ 562 &= 427 \times 1 + 135 \\ (427, 562) &= (135, 427) \\ 427 &= 135 \times 3 + 22 \\ (135, 427) &= (22, 135) \end{aligned}$$

This process could continue, but at this point it is quicker and easy to notice that the two numbers have no common factor other than 1, so $(4361, 9284) = 1$.

b
$$\begin{aligned} 2160 &= 999 \times 2 + 162 \\ (999, 2160) &= (162, 999) \\ 999 &= 162 \times 6 + 27 \\ (162, 999) &= (27, 162) \\ 162 &= 27 \times 6 + 0 \\ (999, 2160) &= 27 \end{aligned}$$

c
$$\begin{aligned} (-372, 762) &= (372, 762) \\ 762 &= 372 \times 2 + 18 \\ (372, 762) &= (372, 18) \\ 372 &= 18 \times 20 + 12 \\ (372, 18) &= (12, 18) \\ 18 &= 12 \times 1 + 6 \\ (12, 18) &= (6, 12) \\ 12 &= 6 \times 2 + 0 \\ (-372, 762) &= 6 \end{aligned}$$

d
$$\begin{aligned} 716\,485 &= 5255 \times 136 + 1805 \\ (716\,485, 5255) &= (1805, 5255) \\ 5255 &= 1805 \times 2 + 1645 \\ (1805, 5255) &= (1805, 1645) \\ 1805 &= 1645 \times 1 + 160 \\ (1805, 1645) &= (160, 1645) \\ 1645 &= 160 \times 10 + 45 \\ (160, 1645) &= (45, 160) \end{aligned}$$

This process could continue, but at this point it is quicker and easy to notice that the two numbers have a highest common factor of 5, so $(716\,485, 5255) = 5$.

4 a Apply the division algorithm to 804 and 2358.

$$\begin{aligned} 2358 &= 804 \times 2 + 750 \\ 804 &= 750 \times 1 + 54 \\ 750 &= 54 \times 13 + 48 \\ 54 &= 48 \times 1 + 6 \\ 48 &= 6 \times 8 \end{aligned}$$

Working backwards with these results,

$$\begin{aligned} 6 &= 54 - 48 \times 1 \\ 6 &= 54 - (750 - 54 \times 13) \\ 6 &= 54 - 750 + 54 \times 13 \\ 6 &= 54 \times 14 - 750 \\ 6 &= (804 - 750 \times 1) \times 14 - 750 \\ 6 &= 804 \times 14 - 750 \times 14 - 750 \\ 6 &= 804 \times 14 - 750 \times 15 \\ 6 &= 804 \times 14 - (2358 - 804 \times 2) \times 15 \\ 6 &= 804 \times 14 - 2358 \times 15 + 804 \times 30 \\ 6 &= 804 \times 44 - 2358 \times 15 \end{aligned}$$

A solution is $x = 44, y = -15$.

The general solution is

$$\begin{aligned} x &= 44 + \frac{2358}{6}t \\ &= 44 + 393t \\ y &= -15 - \frac{804}{6}t \\ &= -15 - 134t,\ t \in \mathbb{Z} \end{aligned}$$

b This is equivalent to $3x + 4y = 1$.

The algorithm is still useful.

$$4 = 3 \times 1 + 1$$
$$1 = 3 \times -1 + 4$$

A solution is $x = -1, y = 1$.

The general solution is

$$x = -1 + 4t$$
$$y = 1 - 3t,\ t \in \mathbb{Z}$$

c $478 = 3 \times -478 + 4 \times 478$

A solution is $x = -478, y = 478$.

The general solution is

$$x = -478 + 4t$$
$$y = 478 - 3t,\ t \in \mathbb{Z}$$

(If you use the algorithm, you will find the solution $x = 1434, y = -956$. Then the general solution can be expressed as

$$x = 1434 + 4t$$

as

$$y = -956 - 3t,\ t \in \mathbb{Z}.)$$

d The algorithm is still useful.

$$-5 = 3 \times -2 + 1$$
$$1 = 3 \times 2 + -5$$
$$1 = 3 \times 2 - 5 \times 1$$
$$38 = 3 \times 76 - 5 \times 38$$

A solution is $x = 76, y = 38$

The general solution is

$$x = 76 + 5t$$
$$y = 38 + 3t$$

This can be simplified. If $t - 15$ is used instead of t, then

$$x = 76 + 5(t - 15)$$
$$= 1 + 5t$$
$$y = 38 + 3(t - 15)$$
$$= -7 + 3t,\ t \in \mathbb{Z}$$

e Apply the division algorithm to 804 and 2688.

$$2688 = 804 \times 3 + 276$$
$$804 = 276 \times 2 + 252$$
$$276 = 252 \times 1 + 24$$
$$252 = 24 \times 10 + 12$$
$$24 = 12 \times 2$$

Working backwards with these results,

$12 = 252 - 24 \times 10$
$12 = 252 - (276 - 252 \times 1) \times 10$
$12 = 252 - 276 \times 10 + 252 \times 10$
$12 = 252 \times 11 - 276 \times 10$
$12 = (804 - 276 \times 2) \times 11 - 276 \times 10$
$12 = 804 \times 11 - 276 \times 22 - 276 \times 10$
$12 = 804 \times 11 - 276 \times 32$
$12 = 804 \times 11 - (2688 - 804 \times 3) \times 32$
$12 = 804 \times 11 - 2688 \times 32 + 804 \times 96$
$12 = 804 \times 107 - 2688 \times 32$

A solution is $x = 107, y = -32$.

The general solution is

$$\begin{aligned} x &= 107 + \frac{2688}{12}t \\ &= 107 + 224t \\ y &= -32 - \frac{804}{12}t \\ &= -32 - 67t,\ t \in \mathbb{Z} \end{aligned}$$

f Apply the division algorithm to 1816 and 2688.

$2688 = 1816 \times 1 + 872$
$1816 = 872 \times 2 + 72$
$872 = 72 \times 12 + 8$
$72 = 8 \times 9$

Working backwards with these results,

$8 = 872 - 72 \times 12$
$8 = 872 - (1816 - 872 \times 2) \times 12$
$8 = 872 - 1816 \times 12 + 872 \times 24$
$8 = 872 \times 25 - 1816 \times 12$
$8 = (2688 - 1816 \times 1) \times 25 - 1816 \times 12$
$8 = 2688 \times 25 - 1816 \times 25 - 1816 \times 12$
$8 = 2688 \times 25 - 1816 \times 37$

A solution is $x = -37, y = 25$.

The general solution is

$$\begin{aligned} x &= -37 + \frac{2688}{8}t \\ &= -37 + 336t \\ y &= 25 - \frac{1816}{8}t \\ &= 25 - 227t, t \in \mathbb{Z} \end{aligned}$$